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Stability and Index of the Meet Game on a Lattice

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Abstract. We study the stability and the stability index of the *meet game form* defined on a meet-semilattice. Given any active coalition structure, we show that the stability index relative to the equilibrium, to the beta core and to the exact core is a function of the Nakamura number, the depth of the semilattice and its gap function.

Keywords: Effectivity Function, Lattice, Stability Index, Equilibrium, Nakamura Number.

JEL Classification: C70, D71 **AMS Classification:** 91A44

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Introduction

The object of this paper is to study the stability of the *meet game form*. Let $(A, \wedge)$ be a meet-semilattice. The n-player meet game form on A is defined as follows: each player chooses $x_i \in A$, the outcome is given by $\mu(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n$. Let $\mathcal{M}$ be any subset of non empty coalitions. Solutions that are considered in this paper are either the β-core, or the exact-core or Nash-like equilibrium where only coalitions in $\mathcal{M}$ are active. Given a solution concept, stability means that for any preference profile, the game form admits at least one such solution, while the stability index is a measure of instability (see [2] for an introduction to this notion). It turns out, that the stability and stability index depend on three parameters: On the side of the players the Nakamura number or $\nu_{\mathcal{M}}$, and on the side of the alternative set, the depth of A or δ_A and the gap function or γ_A.

1 Game forms

1.1 Notations

Throughout this paper we shall consider a finite set $N = \{1, \ldots, n\}$ the elements of which are called *players*, and a finite set $A = \{a_1, \ldots, a_p\}$ the elements of which are called *alternatives*. We make use of the following notational conventions: For any set X, we denote by $\mathcal{P}(X)$ the set of all subsets of X and by $\mathcal{P}_0(X) = \mathcal{P}(X) \backslash \{\emptyset\}$ the set of all non-empty subsets of X. $Q(X)$ (resp. $L(X\ )$) will denote the set of all preorders (resp. linear orders) on X, that is all binary relations on X which are transitive and complete (resp. transitive, complete and antisymmetric). If $R \in Q(X)$ we denote by R° (resp. $R^\sim$) the strict binary relation (resp. the equivalence relation) induced by R on X. Elements of $\mathcal{P}_0(N)$ are called *coalitions*. If $S \in \mathcal{P}_0(N)$ then $N \backslash S$ is denoted S^c. Similarly if $B \in \mathcal{P}(A)$, $A \backslash B$ is denoted B^c. A preference profile (over A) is a map from N to $Q(A)$, so that a preference profile is an element of $Q(A)^N$. For every preference profile $R_N \in Q(A)^N$ and $S \in \mathcal{P}_0(N)$ we put

$$P(a, S, R_N) = \{b \in A \mid \ b\ R_i^\circ\ a, \forall i \in S\}$$

(so that $P(a, S, R_N)$ consists of all the outcomes considered to be strictly better than a by all members of the coalition S), and $P^c(a, S, R_N) = A \backslash P(a, S, R_N)$.

1.2 Game forms and solutions

Let $G = \langle X_1, \ldots, X_n, A, g \rangle$ be a strategic game form. The set of players is $N = \{1, \ldots, n\}$, X_i is the strategy set of players i, $g : \prod_{i \in N} X_i \to A$ is the outcome function, assumed to be surjective. For any $S \in \mathcal{P}_0(N)$ the product $\prod_{i \in S} X_i$ will be denoted X_S. An element $(x_i)_{i \in N} \in X_N$ will be denoted simply x_N and its projection on X_S will be denoted x_S. Given any preference profile $R_N \in Q(A)^N$, the game form G induces a game $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$ with the same strategy spaces and where Q_i is the preorder on X_N defined by: $x_N\, Q_i\, y_N$ if and only if $g(x_N)\, R_i\, g(y_N)$ for $x_N, y_N \in X_N$. We denote this game by $G(R_N)$.

For our solution concepts we shall assume that only some coalitions can form. Any $\mathcal{M} \subset \mathcal{P}_0(N)$ is called an *active coalition structure.* The first solution concept is similar to Nash equilibrium. It has been introduced in [6] (definition 5.1.6):

• A strategy array $x_N \in X_N$ is an $\mathcal{M}$-*equilibrium* of the game $G(R_N)$ if there is no coalition $S \in \mathcal{M}$ and $y_S \in X_S$ such that $g(y_S, x_{S^c})\ R_i^\circ\ g(x_N)$ for all $i \in S$. An alternative a is an $\mathcal{M}$-*equilibrium outcome* of G at R_N if there exists some equilibrium $x_N \in X_N$ of $G(R_N)$ such that $g(x_N) = a$. We denote by $EO(\mathcal{M})(G, R_N)$ the set of all $\mathcal{M}$-equilibrium outcomes of (G, R_N). In particular, when $\mathcal{M} = \mathcal{N} \equiv \{\{1\}, \ldots, \{n\}\}$, an $\mathcal{M}$-equilibrium is a Nash equilibrium. Similarly, when $\mathcal{M} = \mathcal{P}_0(N)$, an $\mathcal{M}$-equilibrium is a strong Nash equilibrium.

The following solutions have been defined respectively in [1] and [4]:

• An alternative a is in the $\mathcal{M}$-*exact core* of (G, R_N) if there is no coalition $S \in \mathcal{M}$ with the following property : for any $z_N \in X_N$ such that $g(z_N) = a$ there exists $y_S \in X_S$ such that $g(y_S, z_{S^c})\ R_i^\circ\ g(z_N)$ for all $i \in S$. Denote by $C_{1,\mathcal{M}}(G, R_N)$ the $\mathcal{M}$-exact core of (G, R_N).

• An alternative a is in the $\mathcal{M}$-β-*core* of (G, R_N) if there is no coalition $S \in \mathcal{M}$ with the following property: for any $z_N \in X_N$, there exists $y_S \in X_S$ such that $g(y_S, z_{S^c})\ R_i^\circ\ a$ for all $i \in S$. Denote by $C_{0,\mathcal{M}}(G, R_N)$, the $\mathcal{M}$-β-core of (G, R_N)

Let Π_r denote the set of all partitions of A with r elements (classes). If $\pi \in \Pi_r$ and $a \in A$ we denote by $\pi(a)$ the class of the partition that contains a. Let $Q_\bullet(\pi)$ be the set of all $R \in Q(A)$ such that whenever $\pi(a) = \pi(b)$ then $aR^\sim b$. We say that G is r-$\mathcal{M}$-*solvable* if $EO(\mathcal{M})(G, R_N) \neq \emptyset$ for all $R_N \in Q_\bullet(\pi)^N$ and all $\pi \in \Pi_r$. *G is r-$\mathcal{M}$-exactly stable* if $C_{1,\mathcal{M}}(G, R_N) \neq \emptyset$ for all $R_N \in Q_\bullet(\pi)^N$ and all $\pi \in \Pi_r$. *G is r-$\mathcal{M}$-β-stable* if $C_{0,\mathcal{M}}(G, R_N) \neq \emptyset$ for all $R_N \in Q_\bullet(\pi)^N$ and all $\pi \in \Pi_r$. We say that G is $\mathcal{M}$-*solvable* if G is r-$\mathcal{M}$-solvable for all $r \geq 1$. Similar definitions can be made for the $\mathcal{M}$-exact core and the $\mathcal{M}$-β-core.

Definition 1.1 The *stability index* of G relatively to the $\mathcal{M}$-equilibrium (resp. $\mathcal{M}$-exact core, resp. $\mathcal{M}$-β-core) is the smallest integer $r \geq 1$ such G is not r-$\mathcal{M}$-solvable (resp. r-$\mathcal{M}$-exactly stable, r-$\mathcal{M}$-β-stable) (with the convention that the index is $+\infty$ if no such integer exists).

The object of this paper is to give necessary and sufficient conditions for stability and determine the stability index of the following game form, called the *meet game form* $\Gamma = \langle X_1, \ldots, X_n, A, \mu\rangle$, where $X_1 = \cdots = X_n = A$, A is a meet-semilattice (precise definitions are given below), and μ is the meet function that is:

$$\mu(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n \quad (x_1 \in A, \ldots, x_n \in A). \tag{1}$$

2 Definitions related to binary relations

For $q \in \mathbb{N}^*$ the set $\{1, \ldots, q\}$ will be denoted $\mathbb{I}_q$. An *interval* of $\mathbb{Z}/q\mathbb{Z}$ is any sequence $(k_1, \ldots, k_r)$ in $\mathbb{Z}/q\mathbb{Z}$, where $r \in \mathbb{I}_{q+1}$ and $k_{s+1} = k_s + 1$ $(s = 1, \ldots, r-1)$. Thus if $r < q+1$, the elements of an interval $(k_1, \ldots, k_r)$ are distinct. When $k = q+1$ we have $k_1 = k_r$ and the interval is said to be *closed.* A *directed graph* or *digraph* is an ordered pair $(A, \searrow)$ where $\searrow$ is a binary relation on A. A couple $(a, b) \in A \times A$ such that

$a \searrow b$ will be called a *step*. Let $q \in \mathbb{N}^*$. A q-*enumeration* of A is an injective mapping $e : \mathbb{Z}/q\mathbb{Z} \to A$. Let e be a q-enumeration of A. An e-*edge* is any ordered pair of the form $v = (e_k, e_{k+1})$ where $k \in \mathbb{Z}/q\mathbb{Z}$. Thus a 1-enumeration e has only one edge (e_1, e_1). Two e-edges v and w are said to be *adjacent* if $v = (e_k, e_{k+1})$ and $w = (e_\ell, e_{\ell+1})$ and $k + 1 = \ell$. An e-*chain* is any sequence $c = (v_1, \ldots, v_r)$ of distinct e-edges such that v_k and v_{k+1} are adjacent ($k = 1, \ldots, r-1$). The *length* of c is the number of its e-edges. It is denoted $|c|$. Since there are no repetition of edges in a chain: $|c| \leq q$. Alternatively, an e-chain is the image by e of some interval of $\mathbb{Z}/q\mathbb{Z}$, with the order induced by e. There are exactly q e-chains with length q, where only the initial vertex differ; we shall identify them all with e. An e-edge is an e-*step* if it is a step. We usually use the same notation for an e-chain (a sequence of e-edges) and the set of its edges. Thus $c \cap c' = \emptyset$ means that c and c' do not have common edges. Let c and c' be two e-chains such that $c' \subset c$. We say that c' is a c-*gap* if , if c' contains no steps and if it is maximal for inclusion in c for this property. If c is an e-chain, we denote by $d(c)$ the number of e-steps in c, and $g(c)$ the number of c-gaps. It is easy to see that $d(c) + g(c) \leq |c|$. For $k \geq 1$, let C_e^k be the set of all e-chains such that $d(c) = k$. We introduce the following numbers related to the graph structure:

$$\delta_A = \max_e d(e) \text{ where } e \text{ describes all the set of } p\text{-enumerations.}$$

$$\gamma_e(k) = \min_{c \in C_e^k} g(c) \text{ with the convention } \gamma_e(k) = +\infty \text{ if } C_e^k = \emptyset.$$

$$\gamma_A(k) = \min_e \gamma_e(k) \text{ where } e \text{ describes all the set of } p\text{-enumerations.}$$

δ_A will be called the *depth* of A, $\gamma_e(.)$ will be called the *gap function* of e and $\gamma_A(.)$ will be call the *gap function* of A. Remark that γ_e and γ_A are increasing functions. By convention $\gamma_e(+\infty) = \gamma_A(+\infty) = +\infty$.

Examples 2.1 (a) Let $A = \{1, \ldots, p\}$ and $a \searrow b$ if and only if $a = b + 1$ (addition in $\mathbb{N}$). Let e be the p-enumeration $e(k) = p - k + 1 \pmod{p}$ then $\delta_A = d(e) = p - 1$. $\gamma_e(k) = 0$ if $1 \leq k < p$, $\gamma_e(k) = +\infty$ if $k \geq p$. $\gamma_A = \gamma_e$.

(b) Let $A = \{1, \ldots, p\}$ ($p \geq 2$) and $a \searrow b$ if and only if $a = b+1$ (addition in $\mathbb{Z}/p\mathbb{Z}$). Let e be the p-enumeration $e(k) = p - k + 1$ then $\delta_A = d(e) = p$. $\gamma_e(k) = 0$ if $1 \leq k \leq p$, $\gamma_e(k) = +\infty$ if $k > p$. $\gamma_A = \gamma_e$.

(c) Let $A = \{1, \ldots, p\}$ ($p \geq 2$) and $a \searrow b$ if and only if $a = p$, $b \neq p$. Let e be the p-enumeration $e(k) = p - k + 1$ then $\delta_A = d(e) = 1$. $\gamma_e(k) = 1$ if $k = 1$, $\gamma_e(k) = +\infty$ if $k > 1$. $\gamma_A = \gamma_e$.

A digraph $(A, \searrow)$ is said to be *acyclic* if for any $q \in \mathbb{N}^*$, any q-enumeration e contains at least one e-gap. A *partially ordered set*, or *poset*, is a pair $(A, \geq)$ where $\geq$ is a binary relation on A that is reflexive, transitive and antisymmetric. To a poset $(A, \geq)$ we shall associate the digraph $(A, >)$ where $x > y$ if and only if $x \geq y$ and $x \neq y$. $(A, >)$ is then an acyclic digraph. A poset is a *meet-semilattice* if any pair $\{x, y\} \subset A$ has an infimum, that is a greatest lower bound, denoted $x \wedge y$. The infimum of any family $(x_1, \ldots, x_k)$ will be denoted $x_1 \wedge \cdots \wedge a_k$.

Definition 2.2 Let $(A, \searrow)$ be a digraph, let e be a q-enumeration and let $\tilde{e}$ be a p-enumeration. $\tilde{e}$ is an *extension* of e if there exists $k \in \mathbb{I}_q$ such that $(e_{k+1}, e_{k+2}, \ldots, e_{k+q})$ (mod q) is an $\tilde{e}$-chain, or equivalently if there is a bijection j from $\mathbb{Z}/q\mathbb{Z}$ onto some interval of $\mathbb{Z}/p\mathbb{Z}$ such that $e = \tilde{e} \circ j$.

Lemma 2.3 *Let* $(A, \searrow)$ *be a digraph. Let* e *be a* q*-enumeration containing some gap* h *and some chain* c*, and let* $k = d(c)$*. There exists some* e*-chain* c' *such that* $d(c') = k$ *and* $c' \cap h = \emptyset$*. Any* e*-chain* c' *such that* $g(c') = \gamma_e(k)$ *leaves some gap in its complement. In particular :* $g(c') < g(e)$*.*

Proof. All the e-steps are in $\bar{h}$, therefore the first assertion. In particular: $g(c) < g(e)$. Any e-chain c' such that $g(c') = \gamma_e(k)$ must leave some gap in his complement, otherwise we would have $g(c) = g(e)$, a contradiction. □

Lemma 2.4 *Let* $(A, \searrow)$ *be a digraph. Let* e *be a* q*-enumeration containing some gap* h *and some chain* c *that do not intersect. Then there exists an extension* $\tilde{e}$ *of* e *such that* c *is an* $\tilde{e}$*-chain.*

Proof. Let $B = A \setminus \{e_1, \ldots, e_q\}$. Then $|B| = p - q$. Let f be any bijection of $\mathbb{I}_{p-q}$ onto B. Without loss of generality let (e_q, e_1) be some e-edge of h. One can define $\tilde{e}$ as follows: $\tilde{e}(\ell) = e(\ell)$ for $\ell \in \mathbb{I}_q$ and $\tilde{e}(q + k) = f(k)$ for any $k \in \mathbb{I}_{p-q}$. It is clear that c is an $\tilde{e}$-chain. □.

Proposition 2.5 *Let* $(A, \searrow)$ *be an acyclic digraph, and let* $k \geq 1$*. Then:*

(i) δ_A *is the maximum of* $d(e)$ *where* e *describes the union of all* q*-enumerations* $(q = 1, \ldots, p)$*.*

(ii) $\gamma_A(k)$ *is the minimum of* $\gamma_e(k)$ *where* e *describes the union of all* q*-enumerations* $(q = 1, \ldots, p)$*.*

Proof of (i). If e is a q-enumeration, such that $d(e)$ achieves the maximum defined in the statement, then, in view of the acyclicity of the digraph and lemma 2.3, there exists some e-chain c of e and some e-gap h such that $d(c) = d(e)$ and h does not interset c. In view of lemma 2.4, there is an extension $\tilde{e}$ of e such that c is a chain of $\tilde{e}$. Since the number of steps of c is the same in e and $\tilde{e}$. This proves (i).

Proof of (ii). If c is some e-chain where e is a q-enumeration, such that $g(c)$ achieves the minimum defined in the statement, then, in view of lemma 2.3, there exists some e-gap h that does not intersect c. In view of lemma 2.4, there is an extension $\tilde{e}$ of e such that c is a chain of $\tilde{e}$. Since the number of steps and gaps in c remain the same, (ii) is proved. □

3 Effectivity structures

Definition 3.1 A *local effectivity function* on (N, A) is a family $E \equiv (E[U],\ U \in \mathcal{P}_0(A))$ where for any $U \in \mathcal{P}_0(A)$, $E[U] : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}_0(A))$ and such that the following conditions are satisfied:

(i) $E[U](\emptyset) = \emptyset$,

(ii) $B \in E[U](S), B \subset B' \Rightarrow B' \in E[U](S)$,

(iii) $U \subset V \Rightarrow E[V](S) \subset E[U](S)$.

A local effectivity function is an *effectivity function* if it does not depend on U. The formula $B \in E[U](S)$ is interpreted as follows: When the current state is in U, coalition

S can adapt its response in order to achieve some state in B. Let $R_N \in Q(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $U \in \mathcal{P}_0(A)$, $S \in \mathcal{P}_0(N)$ such that $a \in U$ and $P(a, S, R_N) \in E[U](S)$. The *core* of E at R_N is the set of undominated alternatives. It is denoted $C(E, R_N)$. We say that E is *r-stable* if $C(E, R_N) \neq \emptyset$ for all $R_N \in Q_\bullet(\pi)^N$ and all $\pi \in \Pi_r$. We say that E *is stable* if E is r-stable for all $r \geq 1$. The *stability index* of E is the minimal integer r such that E is not r-stable (with the convention that this index is $+\infty$ if E is stable). It will be denoted $\sigma(E)$.

Let G be a strategic game form. The *local effectivity function* $E^G_{1,\mathcal{M}}$ *associated to* $(G, \mathcal{M})$ is defined as follows: For $U \in \mathcal{P}_0(A)$, $S \notin \mathcal{M}$: $E^G_{1,\mathcal{M}}[U](S) = \emptyset$, and for $S \in \mathcal{M}$:

$$E^G_{1,\mathcal{M}}[U](S) = \{B \in \mathcal{P}_0(A) | \, \forall x_N \in g^{-1}(U), \exists y_S \in X_S : g(x_{S^c}, y_S) \in B\}$$

The β-effectivity function associated to $(G, \mathcal{M})$ is defined by $E^G_{0,\mathcal{M}}(S) = E^G_{1,\mathcal{M}}[A](S)$ $(S \in \mathcal{P}(N))$.

Lemma 3.2 *The $\mathcal{M}$-exact core (resp. $\mathcal{M}$-β-core) of (G, R_N) coincides with the core of $E^G_{1,\mathcal{M}}$ (resp. $E^G_{0,\mathcal{M}}$) at R_N. Therefore G is r-$\mathcal{M}$-exactly stable (resp. r-$\mathcal{M}$-β-stable) if and only if $E^G_{1,\mathcal{M}}$ (resp. $E^G_{0,\mathcal{M}}$) is r-stable. In particular the stability index relatively to the $\mathcal{M}$-exact core of G is equal to the stability index of $E^G_{1,\mathcal{M}}$.*

Proof. Straightforward. □

Definition 3.3 Let E ba a local effectivity function. An r-tuple $((C_1, B_1, S_1), \ldots, (C_r, B_r, S_r))$ where $r \geq 1$, $C_k \in \mathcal{P}_0(A)$, $B_k \in \mathcal{P}_0(A)$, $S_k \in \mathcal{P}_0(N)$ $(k = 1, \ldots, r)$ is a *dominance configuration* of E if:

(i) $B_k \in E[C_k](S_k)$ $(k = 1, \ldots, r)$.

(ii) $(C_1, \ldots, C_r)$ is a partition of E.

$(C_1, \ldots, C_r)$ is said to be the *basis* of the dominance configuration and r its *length* or *order.*

A dominance configuration $((C_1, B_1, S_1), \ldots, (C_r, B_r, S_r))$ is a *cycle* of E if it satisfies the following property :

(C) For any $\emptyset \neq J \subset \{1, \ldots, r\}$ such that $\cap_{k \in J} S_k \neq \emptyset$, there exists $k \in J$ such that for all $l \in J$: $B_k \cap C_l = \emptyset$.

In the context of effectivity functions cycles have been introduced in [3]. They generalize the Condorcet cycle and play a fundamental role in studying stability. In view of Abdou [2] Theorem 4.4, we have:

Theorem 3.4 *The stability index of a local effectivity function E is equal to the minimal length of a cycle of E (with the convention that this number is $+\infty$ if E has no cycle)*

Finally we need to recall from [5] a classical definition. Let $\mathcal{M}$ be an active coalition structure. A nonempty subset $\mathcal{T} \subset \mathcal{M}$ has the *empty intersection property* if $\cap_{S \in \mathcal{T}} S = \emptyset$. The *Nakamura Number* of $\mathcal{M}$, denoted $\nu_\mathcal{M}$, is the minimum of the cardinality of $\mathcal{T}$ where $\mathcal{T}$ describes all the subsets of $\mathcal{M}$ with the empty intersection property (with the convention that this number is $+\infty$ if no subset of $\mathcal{M}$ has the empty intersection property).

4 The meet game form

In this section $(A, \geq)$ is a meet-semilattice and $\Gamma = \langle X_1, \ldots, X_n, A, \mu \rangle$ is the meet game form (1) defined on A. Γ has the following remarkable property:

Proposition 4.1 *For any $R_N \in Q(A)^N$, an outcome is an $\mathcal{M}$-equilibrium outcome of Γ if and only if it is in the $\mathcal{M}$- exact core of Γ that is :*

$$EO(\mathcal{M})(\Gamma, R_N) = C_{1,\mathcal{M}}(\Gamma, R_N)$$

.

Proof. $EO(\mathcal{M})(\Gamma, R_N) \subset C_{1,\mathcal{M}}(\Gamma, R_N)$ for any game form. In order to prove the opposite inclusion, let $a \notin EO(\mathcal{M})(\Gamma, R_N)$. For any $x = (x_1, \ldots, x_n)$ such that $\mu(x) = a$, there exists some $S_x \in \mathcal{M}$ and y_{S_x} such that $\mu(x_{S_x^c}, y_{S_x}) R^\circ a$ for all $i \in S_x$. The main point is to prove that one can choose some "deviation " $(S_x, y_{S_x^c})$ that do not depend on x. Let S be the coalition corresponding to $\bar{x} = (a, \ldots, a)$ and let $b = \mu(\bar{x}_{S^c}, y_S)$. One has $bR_i^\circ a$ for all $i \in S$. Let $c = \wedge_{i \in S} y_i$. Then $a \wedge c = b$. Clearly $b \neq a$. If $S = N$ then for any x such that $\mu(x) = a$, $\mu(y_N) = b = c$ thus $a \notin C_{1,\mathcal{M}}(\Gamma, R_N)$. If $S \neq N$ then $b < a$. Let $b_S \in A^S$ with all components equal to b. For any x such that $\mu(x) = a$ one has: $b < a \leq \wedge_{j \in S^c} x_j$. It follows that $\mu(x_{S^c}, b_S) = (\wedge_{j \in S^c} x_j) \wedge b = b$. Again $a \notin C_{1,\mathcal{M}}(\Gamma, R_N)$. $\square$

Corollary 4.2 *The meet game form Γ is $\mathcal{M}$- solvable if and only if it is $\mathcal{M}$-exactly stable. The stability index of Γ is the same whether we consider the $\mathcal{M}$- exact core or the $\mathcal{M}$-equilibrium.*

Thus studying stability of the local effectivity function is sufficient not only for $\mathcal{M}$-exact stability of Γ, but also for its $\mathcal{M}$- solvability. Here is its precise description for any $\mathcal{M}$:

Proposition 4.3 *For $a \in A$, one has :*

$$E^\Gamma_{1,\mathcal{M}}[U](S) = \begin{cases} \{B \in \mathcal{P}_0(A) \mid \forall a \in U, \exists b \in B : a \geq b\} & \text{if } S \in \mathcal{M}, S \neq N \\ \mathcal{P}_0(A) & \text{if } S \in \mathcal{M}, S = N \\ \emptyset & \text{if } S \notin \mathcal{M} \end{cases}$$

Proof. Since $E^\Gamma_{1,\mathcal{M}}[U](S) = \cap_{a \in U} E^\Gamma_{1,\mathcal{M}}[a](S)$, it is enough to prove the formula for $E^\Gamma_{1,\mathcal{M}}[a](S)$ $(a \in A)$. That $E^\Gamma_{1,\mathcal{M}}[a](N) = \mathcal{P}_0(A)$ is straightforward. Let $S \in \mathcal{P}_0(N)$, $S \neq N$ and let $B \in \mathcal{P}_0(A)$ and $b \in A$ such that $b \in B$ and $b \leq a$. Let $b_S \in A^S$ with all components equal to b. For any x such that $\mu(x) = a$, $b \leq a \leq (\wedge_{j \in S^c} x_j)$. Thus $\mu(x_{S^c}, b_S) = (\wedge_{j \in S^c} x_j) \wedge b = b$. Therefore $B \in E^\Gamma_{1,\mathcal{M}}[a](S)$. Conversely if $B \in E^\Gamma_{1,\mathcal{M}}[a](S)$, then in particular taking $x = (a, \ldots, a) \in A^N$ there exists $y_S \in A^S$ such that $\mu(x_S, y_{S^c}) \in B$. Since $\mu(x_S, y_{S^c}) \leq a$, the proof is complete. $\square$.

We conclude this section, by a statement of the main results of the paper:

Theorem 4.4 *For any $\mathcal{M}$, the meet game form Γ is $\mathcal{M}$-β-stable. Γ is $\mathcal{M}$-exactly stable (and therefore $\mathcal{M}$-solvable) if and only if either $N \notin \mathcal{M}$ or $\delta_A < \nu_{\mathcal{M}}$.*

In particular the meet game form is Nash solvable. If $n \geq 2$ and $\mathcal{M} = \mathcal{P}_0(N)$, then $\nu_{\mathcal{M}} = 2$. Thus the meet game form is strongly solvable if and only if $\delta_A = 1$.

Theorem 4.5 *If $N \in \mathcal{M}$, the stability index of the meet game form relatively to the $\mathcal{M}$-exact core (and therefore $\mathcal{M}$ -equilibrium) is equal to: $\nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$.*

In particular, if $n \geq 2$ and $\mathcal{M} = \mathcal{P}_0(N)$, the strong Nash stability index of the meet game form is equal to $\gamma_A(\nu_{\mathcal{M}}) + 3$. In the next section, we give a proof of both theorems in a more general framework.

5 Stability and Index of the meet game form

In this section we assume that $(A, \searrow)$ is an acyclic digraph. We shall write $(a \searrow\!\!\searrow b)$ if $(a \searrow b)$ or $(a = b)$. For any $\emptyset \neq \mathcal{M} \subset \mathcal{P}_0(N)$, we consider the local effectivity function $E_{\mathcal{M}}$ defined as follows: For $U \in \mathcal{P}_0(A)$:

$$E_{\mathcal{M}}[U](S) = \begin{cases} \{B \in \mathcal{P}_0(A) \mid \forall a \in U, \exists b \in B : a \searrow\!\!\searrow b\} & \text{if } S \in \mathcal{M}, S \neq N \\ \mathcal{P}_0(A) & \text{if } S \in \mathcal{M}, S = N \\ \emptyset & \text{if } S \notin \mathcal{M} \end{cases}$$

The corresponding effectivity function is defined by $E_{0,\mathcal{M}}[U](S) = E_{0,\mathcal{M}}(S) = E_{\mathcal{M}}[A](S)$ $(S \in \mathcal{P}(N))$. Let A_0 be the set of minimal elements of $(A, \searrow\!\!\searrow)$: $x \in A_0$ if and only if there is no $y \in A$ such that $x \searrow y$. Since A is finite and $(A, \searrow)$ acyclic, $A_0 \neq \emptyset$. It is then easy to see that, for any $S \in \mathcal{M}$, $S \neq N$ any $B \in E_{0,\mathcal{M}}(S)$ contains A_0. In the case where $\searrow\!\!\searrow$ is a poset, the converse is also true: $B \in E_{0,\mathcal{M}}(S)$ if and only if $A_0 \subset B$.

Lemma 5.1 *Let $(U_1, \dots, U_r)$ be a partition of A and let $(B_1, \dots, B_r)$ be a family of nonempty subsets of A. Then there exists a subset $I = \{k_1, \dots, k_s\}$ where $1 \leq s \leq r$ such that $B_{k_j} \cap U_{k_{j+1}} \neq \emptyset$ $(j = 1, \dots, s)$ (mod s).*

Proof. Let $\mathcal{I}$ be the set of nonempty subsets $I \in \mathbb{I}_r$ such that for any $k \in I$ there exists $\ell \in I$ such that $B_k \cap C_\ell \neq \emptyset$. Clearly $\mathbb{I}_r \in \mathcal{I}$. Let I_0 be a minimal set for inclusion in $\mathcal{I}$. For any $k \in I_0$ put $\theta(k)$ one of the indices $l \in I_0$ such that $B_k \cap U_\ell \neq \emptyset$. Take $k_1 \in I_0$ arbitrary and put $k_{j+1} = \theta(k_j)$ $j = 1, 2, \ldots$.. By minimality of I_0, the sequence $(k_1, \dots, k_s)$ is composed of distinct indices and $k_{s+1} = k_1$. □

Theorem 5.2 *$E_{0,\mathcal{M}}$ is stable for any $\mathcal{M}$.*

Proof. Assume that $C(E_{0,\mathcal{M}}, R_N) = \emptyset$ for some $R_N \in Q(A)^N$. Let $x_0 \in A_0$. Then $P(x_0, S, R_N) \in E_{0,\mathcal{M}}(S)$ for some $S \in \mathcal{M}$). In view of the remark preceding Lemma 5.1, we cannot have $S \neq N$: indeed $x_0 \in A_0$ and $x_0 \notin P(x_0, S, R_N)$. it follows that $S = N$. Therefore $N \in \mathcal{M}$. Moreover, one can construct by induction a sequence $x_0, \dots, x_{t+1}$ such that x_k is Pareto dominated by x_{k+1} for $k = 0, \dots, t-1$ and x_{t+1} not Pareto dominated. Two consequences follow: (1) $x_{t+1} \in P(x_0, N, R_N)$ and (2): there exists some $S \in \mathcal{M}$, $S \neq N$ such that $P(x_{t+1}, S, R_N) \in E_{0,\mathcal{M}}(S)$. Since $x_0 \in A_0 \subset P(x_{t+1}, S, R_N)$ we have $x_0 \in P(x_{t+1}, S, R_N)$. The latter contradicts (1). □

Theorem 5.3 *$E_{\mathcal{M}}$ is stable if and only if either $N \notin \mathcal{M}$ or $\delta_A < \nu_{\mathcal{M}}$.*

Proof. Assume that $E_{\mathcal{M}}$ is not stable. Let $R_N \in Q(A)^N$ be such that $C(E_{\mathcal{M}}, R_N)$ is empty. Put $A = \{a_1, \ldots, a_p\}$. For any $k \in \mathbb{I}_p$, there exists $b_k \in A$ and $S_k \in \mathcal{M}$ such that $\{b_k\} \in E_{\mathcal{M}}(S_k)$, and $b_k R_i^\circ a_k$ for all $i \in S_k$. Let $U_k = \{a_k\}$ and $B_k = \{b_k\}$. By Lemma 5.1, there exists a subset $I = \{k_1, \ldots, k_s\}$ where $1 \leq s \leq p$ such that $b_{k_j} = a_{k_{j+1}}$ $(j = 1, \ldots, s)$ (mod s). Let $e(j) = b_{k_j}$ for all $j \in \mathbb{I}_s$. Let $J = \{k \in \mathbb{I}_p \mid S_k \neq N\}$. For all $k \in J$, $a_k \searrow b_k$. Then e is an s-enumeration, such that $e(j-1) \searrow e(j)$ if $k_j \in J$. It follows first that I is not a subset of J, for otherwise e would be a cycle for the binary relation $\searrow$ and the latter is acyclic. Therefore there exists $i \in I$ such that $S_i = N$, hence $N \in \mathcal{M}$. It follows also that $|J \cap I| \leq d(e) \leq \delta_A$. On the other hand since $e(j) R_i^\circ e(j-1)$ for all $i \in S_{k_j}$ (mod s) we have $\cap_{k \in J \cap I} S_k = \cap_{k \in I} S_k = \emptyset$, so that $\nu_{\mathcal{M}} \leq |J \cap I|$. We conclude that $\nu_{\mathcal{M}} \leq \delta_A$.

Conversely, assume that $N \in \mathcal{M}$ and $\nu_{\mathcal{M}} \leq \delta_A$. Let e be a p-enumeration such that $d(e) = \delta_A$. Let J be the set of indices $k \in \{1, \ldots, p\}$ such that $e_k \searrow e_{k+1}$ (mod p). Then $|J| = \delta_A$. Let $I \subset J$ such that $|I| = \nu_{\mathcal{M}}$. Let $(T_k, k \in I)$ be a family of elements of $\mathcal{M}$ such that $\cap_{k \in I} T_k = \emptyset$. We consider the n-tuple $F = ((U_1, B_1, S_1), \ldots, (U_p, B_p, S_p))$ defined as follows: For $k \in \{1, \ldots, p\}$, $U_k := \{e_k\}$, $B_k := \{e_{k+1}\}$ (mod p); if $k \in I$, $S_k := T_k$ and if $k \notin I$, $S_k := N$. Since $B_k \in E[U_k](S_k)$ for all $k \in \mathbb{I}_p$, F is a dominance configuration. We now show that this configuration verifies property (C) of definition 3.3. Let K be any subset of $\{1, \ldots, p\}$ such that $\cap_{k \in K} S_k \neq \emptyset$. There exists some $k_0 \in I$ such that $k_0 \notin K$. Let k_1 be the first index in K that appears after k_0 (mod p). $U_{k_1} = \{e_{k_1}\}$ is such that $U_{k_1} \cap (\cup_{k \in K} B_k) = \emptyset$. Thus F is a cycle. □

Theorem 5.4 *Assume $N \in \mathcal{M}$. We have the equality: $\sigma(E_{\mathcal{M}}) = \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$.*

Proof. We first consider the particular case where $\nu_{\mathcal{M}} > \delta_A$. In view of Theorem 5.3, $E_{\mathcal{M}}$ is stable, so that $\sigma = +\infty$. If $\nu_{\mathcal{M}} = +\infty$ then the equality is verified. If $\nu_{\mathcal{M}} < +\infty$ then by definition, since $\nu_{\mathcal{M}} > \delta_A$, one has $\gamma_A(\nu_{\mathcal{M}}) = +\infty$. Again the equality is verified.

Assume that $\nu_{\mathcal{M}} \leq \delta_A$. Let $F = ((U_1, B_1, S_1), \ldots, (U_r, B_r, S_r))$ be a cycle. We are going to prove $r \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$. In view of the structure of $E_{\mathcal{M}}$ there exists $\varphi : A \to A$ with the following properties: (1) $x \in U_k \Rightarrow \varphi(x) \in B_k$ and (2) $T_k \neq N \Rightarrow x \searrow \varphi(x)$. Since the set of minimal elements A_0 is non empty we choose $x_1 \in A_0$. We construct a sequence (x_k) in A by induction by putting $x_{k+1} = \varphi(x_k)$ $k = 1, 2, \ldots$, and a sequence (t_k) in $\mathbb{I}_r$ by defining t_k as the unique element in $\mathbb{I}_r$ such that $x_k \in U_{t_k}$. Let $k_1 \in \mathbb{N}^*$ be the smallest integer such there exists $k_2 \in \mathbb{N}^*$, $k_2 > k_1$ and $t_{k_1} = t_{k_2}$. Clearly $(t_1, \ldots, t_{k_2-1})$ are all distinct. Therefore $k_2 - 1 \leq r$. We distinguish 4 cases:

Case 1. $k_1 > 1$ and $x_{k_2} \neq x_{k_1}$. We put $c = (x_{k_1}, \ldots, x_{k_2})$ and $e = (x_{k_1}, \ldots, x_{k_2}, x_{k_1})$. e is a q-enumeration, c is an e-chain and $q = k_2 - k_1 + 1$. Thus $q \leq (r+1) - 2 + 1 = r$.

Case 2. $k_1 > 1$ and $x_{k_2} = x_{k_1}$. We put $c = e = (x_{k_1}, \ldots, x_{k_2-1}, x_{k_2})$ this is a q-enumeration with $q \leq r - 1$.

Case 3. $k_1 = 1$ and $x_{k_2} \neq x_{k_1}$. We put $c = (x_{k_1}, \ldots, x_{k_2})$ and $e = (x_{k_1}, \ldots, x_{k_2}, x_{k_1})$. e is a q-enumeration, c is an e-chain and $q \leq (r+1) - 1 + 1 = r + 1$. Since $x_{k_1} = x_1 \in A_0$, (x_{k_1}, x_{k_1+1}) is not a step.

Case 4. $k_1 = 1$ and $x_{k_2} = x_{k_1}$. We put $c = e = (x_{k_1}, \ldots, x_{k_2-1}, x_{k_2})$ this is a q-enumeration with $q \leq r$.

First we establish a lower bound on the depth of c. Precisely we prove:

Claim. $d(c) \geq \nu_{\mathcal{M}}$.

Prove of the claim. Put $I = \{k_1, \ldots, k_2 - 1\}$, $J = \{k \in I \mid S_{t_k} \neq N\}$. We claim that: $\cap_{k \in J} S_{t_k} = \emptyset$. The proof is by contradiction: Assume that $\cap_{k \in I} S_{t_k} = \cap_{k \in J} S_{t_k} \neq \emptyset$, then by property (C) of cycles there exists $\ell \in I$ such that for all $k \in I : U_{t_\ell} \cap B_{t_k} = \emptyset$. If $\ell \neq k_1$, $x_\ell \in U_{t_\ell}$ and $x_\ell \in B_{t_{\ell-1}}$, a contradiction. If $\ell = k_1$, then in cases 2 and 4, $x_{k_1} \in U_{t_{k_1}}$ and $x_{k_1} \in B_{t_{k_2-1}}$, a contradiction, and in cases 1 and 3, we have $U_{t_{k_1}} = U_{t_{k_2}}$ $x_{k_2} \in U_{t_{k_2}}$ and $x_{k_2} \in B_{t_{k_2-1}}$, again a contradiction. Thus we proved $\cap_{k \in J} S_{t_k} = \emptyset$. It follows that $|J| \geq \nu_{\mathcal{M}}$. Put $v_k = (x_k, x_{k+1})$ $(k \in I)$. For any $k \in J$, v_k is a step. Therefore $d(c) \geq |J|$. Thus $d(c) \geq \nu_{\mathcal{M}}$, and our claim is proved.

Now, we establish a lower bound on the number of gaps in e and conclude by the desired inequality.

Cases 2 and 4. Here $c = e$. Using lemma 2.3 for the first inequality and monotonicity of γ_A for the third, one has:

$$g(e) \geq \gamma_e(d(e)) + 1 \geq \gamma_A(d(e) + 1 \geq \gamma_A(\nu_{\mathcal{M}}) + 1$$

$$q = |e| \geq d(e) + g(e) \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$$

Moreover $r \geq q$, therefore : $r \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$.

Cases 1 and 3. Here c is an e-chain, $c = e \setminus (x_{k_2}, x_{k_1})$.

$$g(c) \geq \gamma_e(d(c)) \geq \gamma_A(d(c) \geq \gamma_A(\nu_{\mathcal{M}}).$$

Case 1. $|c| \geq d(c) + g(c) \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}})$

$$r \geq q = |c| + 1 \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1.$$

Case 3. Here $v_{k_1} = (x_{k_1}, x_{k_1+1})$ is not a step. Let $c' = c \setminus \{v_{k_1}\}$. We have $d(c') = d(c)$.

$$|c| - 1 = |c'| \geq d(c') + g(c') = d(c) + g(c') \geq d(c) + \gamma_e(d(c)$$

$$|c| \geq d(c) + \gamma_e(d(c) + 1 \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1.$$

Since $q = |c| + 1$ and $r \geq q - 1$, we have: $r \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$.

In conclusion we have in all cases the inequality $r \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$. Thus $\sigma(E_{\mathcal{M}}) \geq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$. This ends the first part of the proof.

Conversely let $r = \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$. We are going to construct a cycle of order r. Let e be a p-enumeration and let c be an e-chain such that $d(c) = \nu_{\mathcal{M}}$ and $g(c) = \gamma_A(\nu_{\mathcal{M}})$. It follows that $\nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) \leq |c|$. In view of lemma 2.3, $\bar{c}$, the complement of c in e contains at least an edge that is not a step. Without loss of generality let $c = (e_1, \ldots, e_q)$ where $q \leq p$. Since $r - 1 = d(c) + g(c)$, we can write c as a sequence of e-steps and e-gaps $(h_1, \ldots, h_{r-1})$. Moreover we put $h_r = (e_q, \ldots, e_p, e_1)$. Let J be the set of indices $k \in \{1, \ldots, r-1\}$ such that h_k is a step. Let $(T_k, k \in J)$ a family of elements in $\mathcal{M} \setminus \{N\}$ such that $\cap_{k \in J} T_k = \emptyset$. Let $f(h_k)$ (resp. $[h_k]$) be the final node (resp. the set of nodes) of h_k ($k \in \mathbb{I}_r$). In particular: $f(h_r) = e_1$. Let $U_k = [h_k] \setminus \{f(h_k)\}$, $B_k = \{f(h_k)\}$ $(k \in \mathbb{I}_r)$. Let $S_k = T_k$ for all $k \in J$ and $S_k = N$ for all $k \in \mathbb{I}_r \setminus J$. We claim that $F = ((U_1, B_1, S_1), \ldots, (U_r, B_r, S_r))$ is a cycle of $E_{\mathcal{M}}$. The only point that we need to verify is property (C) of definition 3.3. Let $K \subset \mathbb{I}_r$ such that $\cap_{k \in K} \neq \emptyset$. There exists some $k_0 \in J \setminus K$. Let ℓ be the first index that comes after k_0 (mod r) such that $\ell \in K$. One has $U_\ell \cap B_k = \emptyset$ for all $k \in K$. This shows that we have a cycle of order r. This shows that $\sigma(E_{\mathcal{M}}) \leq \nu_{\mathcal{M}} + \gamma_A(\nu_{\mathcal{M}}) + 1$. The proof is complete. $\square$

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